Concentration-diffusion effects in viscous incompressible flows*

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Abstract

Given a finite sequence of times $0 < t_1 < \cdots < t_N$, we construct an example of a smooth solution of the free nonstationnary Navier–Stokes equations in $\mathbb{R}^d$, $d = 2, 3$, such that: (i) The velocity field $u(x,t)$ is spatially poorly localized at the beginning of the evolution but tends to concentrate until, as the time t approaches t_1, it becomes well-localized. (ii) Then u spreads out again after t_1, and such concentration-diffusion phenomena are later reproduced near the instants t_2, t_3, ...

1 Introduction

One of the most important questions in mathematical Fluid Mechanics, which is still far from being understood, is to know whether a finite energy, and initially smooth, nonstationnary Navier–Stokes flow will always remain regular during its evolution, or can become turbulent in finite time.

As a first step toward the understanding of possible blow-up mechanisms, it is interesting to exhibit examples of smooth and decaying initial data such that, even if the corresponding solutions remain regular for all time, "something strange" happens around a given point (x_0, t_0) in space-time. This is the goal of the present paper.

Our main result is the construction of a class of (smooth) solutions to the incompressible Navier–Stokes equations such that, in the absence of any external forces, the motion of the fluid particles tends to be more concentrated around x_0, as the time t approaches t_0. This corresponds precisely to the qualitative behavior that one would expect in the presence of a singularity, even though such "concentration of the motion" is not strong enough to imply their formation.

1.1 Statement of the main result

For a fluid filling the whole space $\mathbb{R}^d$, $d \geq 2$, the Navier–Stokes system can be written as

$$\begin{cases} \partial_t u + \mathbb{P}\nabla \cdot (u \otimes u) = \Delta u \\ \nabla \cdot u = 0 \\ u(x,0) = a(x), \end{cases}$$

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where $u = (u_1, \ldots, u_d)$, a is a divergence-free vector field in $\mathbb{R}^d$ and $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\mathrm{div}$ is the Leray-Hopf projector.

Because of the presence of the non-local operator $\mathbb{P}$ a velocity field that is spatially well localized (say, rapidly decaying as $|x| \to \infty$) at the beginning of the evolution, in general, will immediately spread out. A sharp description of this phenomenon is provided by the two estimates (2) below. In order to rule out the case of somewhat pathological flows (such as two dimensional flows with radial vorticity, or the three-dimensional flows described in [3], which behave quite differently as $|x| \to \infty$ if compared with generic solutions), we will restrict our attention to data satisfying the following mild *non-symmetry* assumption (for $j, k = 1, \ldots, d$):

$$\exists\, j \neq k\colon \int (a_j a_k)(x)\,dx \neq 0, \qquad \text{or} \qquad \int a_j(x)^2\,dx \neq \int a_k(x)^2\,dx. \tag{1}$$

Then, for sufficiently fast decaying data, we have, for $|x| \geq \frac{C}{\sqrt{t}}$ (see [4]):

$$\eta_1(t)|x|^{-d-1} \leq |u(x,t)| \leq \eta_2(t)|x|^{-d-1}, \qquad \tfrac{x}{|x|} \in \mathbb{S}^{d-1}\backslash\Sigma_a, \tag{2}$$

these estimates being valid during a small time interval $t \in (0, t_1)$. Here $C, t_1 > 0$ and η_1 and η_2 are positive functions, independent on x, behaving like $\sim c_j\, t$ as $t \to 0$ $(j = 1, 2)$. Moreover, $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$ and the subset Σ_a of $\mathbb{S}^{d-1}$ represents the directions along which the lower bound may fail to hold: the result of [4] tells us that Σ_a can be taken of arbitrarily small surface measure on the sphere. In other words, *the lower bound holds true in quasi-all directions*, whereas the upper bound is valid along all directions.

Moreover, the upper bound will hold during the whole lifetime of the strong solution u (see [12], [13]), whereas the lower bound is valid, *a priori*, only during a very short time interval. The main reason for this is that the matrix $(\int u_j u_k(x,t)\,dx)$ is non-invariant during the Navier–Stokes evolution, in a such way that even if the datum satisfies (1), it cannot be excluded that at later times the solution features some kind of creation of symmetry, yielding to a better spatial localization and, after $t > t_1$, to an improved decay as $|x| \to \infty$. (We refer *e.g.* to [2], [3], [7], [8], for the connection between the symmetry and the decay of solutions).

The purpose of this paper is to show that this indeed can happen. We construct an example of a solution of the Navier–Stokes equations, with datum $a \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class), $d = 2, 3$, such that the lower bound

$$|u(x,t)| \geq \eta_1(t)|x|^{-d-1}, \tag{3}$$

holds in some interval $(0, t_1)$, but then brakes down at t_1, where a *stronger upper bound* can be established. This means that the motion of the fluid concentrates around the origin at such instant. Then the lower bound (3) will hold true again after t_1, until it will break down once more at a time $t_2 > t_1$. This diffusion-concentration effect can be repeated an arbitrarily large number of times.

More precisely, we will prove the following theorem.

Theorem 1 *Let $d = 2, 3$, let $0 < t_1 < \cdots < t_N$ be a finite sequence and $\epsilon > 0$. Then there exist a divergence-free vector field $a \in \mathcal{S}(\mathbb{R}^d)$ and two sequences $(t'_1, \ldots, t'_N)$ and $(t^*_1, \ldots, t^*_N)$ such that the corresponding unique strong solution $u(x,t)$ of the Navier–Stokes system satisfies, for all $i = 1, \ldots, N$ and all $|x|$ large enough, the pointwise the lower bound*

$$|u(x, t'_i)| \geq c_\omega |x|^{-d-1},$$

and the stronger upper bound

$$|u(x,t_i^*)| \le C|x|^{-d-2},$$

for a constant $C>0$ independent on x and a constant c_ω independent on $|x|$, but possibly dependent on the projection $\omega = \frac{x}{|x|}$ of x on the sphere, and such that $c_\omega > 0$ for a.e. $\omega \in \mathbb{S}^{d-1}$. Moreover, t_i' and t_i^* can be taken arbitrarily close to t_i:

$$|t_i' - t_i| < \epsilon \qquad \text{and} \qquad |t_i^* - t_i| < \epsilon, \qquad \text{for } i = 1,\ldots,N.$$

Remark 1 The initial datum can be chosen of the form $a = \mathrm{curl}(\psi)$, where ψ is a linear combination of dilated and modulated of a single function (or vector field, if $d=3$) $\phi \in \mathcal{S}(\mathbb{R}^d)$, with compactly supported Fourier transform.

Roughly speaking, our construction works as follows: we look for an initial datum of the form $a = \mathrm{curl}(\psi)$, where

$$\psi(x) = \sum_{j=1}^{d(N+1)} \lambda_j\, \delta^{d/2}\phi(\delta x)\cos(\alpha_j \cdot x).$$

The unknown vector $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_{d(N+1)}) \in \mathbb{R}^{d^2(N+1)}$ of all the phases $\alpha_j \in \mathbb{R}^d$ will be assumed to belong to a suitable subspace $V \subset \mathbb{R}^{d^2(N+1)}$ of dimension $d(N+1)$ in order to ensure, *a priori*, some nice geometrical properties of the flow. Such geometric properties consist of a kind of rotational symmetry, similar to that considered in [2], but less stringent. In this way, the problem can be reduced to the study of the zeros of the real function

$$t \mapsto \int_0^t \int (u_1 u_2)(x,s)\,dx\,ds.$$

By an analyticity argument, this in turn is reduced to the study of the sign of the function

$$t \mapsto \int_0^t \int (e^{s\Delta}a_1 e^{s\Delta}a_2)(x,s)\,dx\,ds.$$

This last problem is finally reduced to a linear system that can be solved with elementary linear algebra.

The spatial decay at infinity of the velocity field is known to be closely related to special algebraic relations in terms of the moments $\int x^\alpha \mathrm{curl}(u)(x,t)\,dx$ of the vorticity $\mathrm{curl}(u)$ of the flow, see [6]. Thus, one could restate the theorem in an equivalent way in terms of identities between such moments for different values of $\alpha \in \mathbb{N}^d$, which are satisfied at the time t_i^* but brake down when $t = t_i'$.

1.2 A concentration effect of a different nature

The concentration-diffusion effects described in Theorem 1 genuinely depend on the very special structure of the nonlinearity $\mathbb{P}\nabla\cdot(u\otimes u)$, more than to the presence of the Δu term. Even though this result is not known for inviscid flows yet, it can be expected that a similar property should be observed also for the Euler equation.

On the other hand, the Laplace operator, commonly associated with diffusion effects, can be responsible also of concentration phenomena, of a different nature. For example, it can happen that $a(x)$ is a *non decaying* (or very slowly decaying) vector field, but such that the unique strong solution $u(x,t)$ of the Navier–Stokes system have a quite fast pointwise decay as $|x|\to\infty$ (say, $\sim |x|^{-d-1}$). This is typically the case when a has rapidly increasing oscillations in the far field. We will discuss this issue in Section 3. Though elementary, the examples of flows presented in that section have some interest, being closely related to a problem addressed by Kato about strong solutions in $L^p(\mathbb{R}^d)$ when $p<d$, in his well known paper [9].

1.3 Notations

Troughout the paper, if $u=(u_1,\dots,u_d)$ is a vector field with components in a linear space X, we will write $u\in X$, instead of $u\in X^d$. We will adopt a similar convention for the tensors of the form $u\otimes u$. We denote with $e^{t\Delta}$ the heat semigroup.

Let $B(0,1)$ be the unit ball in $\mathbb{R}^d$ and $\phi\in\mathcal{S}(\mathbb{R}^d)$ a function satisfying

$$\widehat\phi\in C_0^\infty(\mathbb{R}^d),\quad \operatorname{supp}\widehat\phi\subset B(0,1),\quad \widehat\phi \text{ radial},\quad \widehat\phi\ge 0,\quad \int|\widehat\phi|^2=1/d. \tag{4}$$

Our definition for the Fourier transform is $\widehat\phi(\xi)=\int\phi(x)e^{-i\xi\cdot x}\,dx$. Then we set

$$\widehat{\phi^\delta}(\xi)=\frac{\widehat\phi(\xi/\delta)}{\delta^{d/2}},\qquad \delta>0. \tag{5}$$

Next we define the orthogonal transformation $\widetilde{\ }\colon\mathbb{R}^d\to\mathbb{R}^d$, by

$$\begin{aligned}
\widetilde\alpha&=(\alpha_2,\alpha_1), &&\text{if } \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2,\\
\widetilde\alpha&=(\alpha_2,\alpha_3,\alpha_1), &&\text{if } \alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3.
\end{aligned} \tag{6}$$

We define the $\operatorname{curl}(\cdot)$ operator by

$$\operatorname{curl}\psi=(-\partial_2,\partial_1)\psi,\qquad \text{if } \psi\colon\mathbb{R}^2\to\mathbb{R}$$

and by

$$\operatorname{curl}\psi=\begin{pmatrix}\partial_2\psi_3-\partial_3\psi_2\\ \partial_3\psi_1-\partial_1\psi_3\\ \partial_1\psi_2-\partial_2\psi_1\end{pmatrix}\qquad \text{if } \psi\colon\mathbb{R}^3\to\mathbb{R}^3.$$

The notation $f(x,t)=\mathcal{O}_t(|x|^{-\alpha})$ as $|x|\to\infty$ means that f satisfies, for large $|x|$, a bound of the form $|f(x,t)|\le A(t)|x|^{-\alpha}$, for some function A locally bounded in $\mathbb{R}^+$.

We shall make use of the usual Kronecker symbol, $\delta_{j,k}=1$ or 0, if $j=k$ or $j\neq k$.

2 Nonlinear concentration-diffusion effects

2.1 The analyticity of the flow map

In this subsection we recall a few well known facts.

Let B be the Navier–Stokes bilinear operator, defined by

$$B(u,v)(t)\equiv-\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(s)\,ds.$$

Then the Navier–Stokes equations can be written in the following integral form

$$u = u_0 + B(u,u), \qquad u_0 = e^{t\Delta}a, \qquad \mathrm{div}(a) = 0. \tag{7}$$

Even though in the sequel we will only deal with "concrete" functional spaces, the problematic is better understood in an abstract setting: we will present it as formulated in the paper by P. Auscher and Ph. Tchamitchian [1]. Let $\mathcal{F}$ be a Banach space, $u_0 \in \mathcal{F}$ and let $B\colon \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ be a continuous bilinear operator, with operator norm $\|B\|$.

Let us introduce the nonlinear operators $T_k\colon \mathcal{F}\to\mathcal{F}$, $k = 1,2\ldots$, defined through the formulae

$$\begin{aligned} T_1 &= \mathrm{Id}_{\mathcal{F}} \\ T_k(v) &\equiv \sum_{l=1}^{k-1} B(T_l(v), T_{k-l}(v)), \quad k \geq 2. \end{aligned}$$

Then the following estimate holds:

$$\|T_k(u_0)\|_{\mathcal{F}} \leq \frac{C}{\|B\|} k^{-3/2} \big(4\,\|B\|\;\|u_0\|_{\mathcal{F}}\big)^k.$$

Moreover, T_k is the restriction to the diagonal of $\mathcal{F}^k = \mathcal{F}\times\cdots\times\mathcal{F}$ of a k-multilinear operator $:\mathcal{F}^k\to\mathcal{F}$.

Under the smallness assumption

$$\|u_0\|_{\mathcal{F}} \leq 1/(4\|B\|), \tag{8}$$

the series

$$\Psi(u_0) \equiv \sum_{k=1}^{\infty} T_k(u_0), \tag{9}$$

is absolutely convergent in $\mathcal{F}$ and its sum $\Psi(u_0)$ is a solution of the equation $u = u_0 + B(u,u)$. Furthermore, $\Psi(u_0)$ is the only solution in the closed ball $\overline{B_{\mathcal{F}}}(0, \frac{1}{2\|B\|})$. The proof of these claims would be a straightforward application of the contraction mapping theorem under the more restrictive condition $\|u_0\|_{\mathcal{F}} < 1/(4\|B\|)$. The proof of the more subtle version stated here can be found in [1], [11].

Coming back to Navier–Stokes, in the proof of Theorem 1 we will need to write the solution of the Navier–Stokes system as $u = \Phi(a)$, where

$$\Phi(a)(t) \equiv \sum_{k=1}^{\infty} T_k(e^{t\Delta}a), \tag{10}$$

the series being absolutely convergent in $\mathcal{C}([0,\infty), L^2(\mathbb{R}^d))$. There are several ways to achieve this, and one of the simplest (which goes through in all dimension $d \geq 2$) is the following: we choose $\mathcal{F}$ as the space of all functions f in $\mathcal{C}([0,\infty), L^2(\mathbb{R}^d))$, such that $\|f\|_{\mathcal{F}} < \infty$, where

$$\|f\|_{\mathcal{F}} \equiv \operatorname{ess\,sup}_{x,t}(1+|x|)^{d+1}|f(x,t)| \;\; + \;\; \operatorname{ess\,sup}_{x,t}(1+t)^{(d+1)/2}|f(x,t)|. \tag{11}$$

The bicontinuity of the bilinear operator B is easily proved in this space $\mathcal{F}$ (see [12]). Indeed, one can prove this only using the well known scaling relations and pointwise estimates on the kernel $F(x,t)$ of the operator $e^{t\Delta}\mathbb{P}\mathrm{div}$:

$$F(x,t) = t^{-(d+1)/2} F(x/\sqrt{t}, 1), \qquad |F(x,1)| \le C(1+|x|)^{-d-1}.$$

We can conclude that there is a constant $\eta_d > 0$, depending only on d, such that if

$$\|e^{t\Delta}a\|_{\mathcal{F}} < \eta_d \tag{12}$$

then there is a solution $u = \Phi(a) \in \mathcal{F}$ of the Navier–Stokes equations such that the series (10) is absolutely convergent in the $\mathcal{F}$-norm. The absolute convergence of such series in $\mathcal{C}([0,\infty), L^2(\mathbb{R}^d))$ is then straightforward under the smallness assumption (12).

The finiteness of $\|e^{t\Delta}a\|_{\mathcal{F}}$ can be ensured, *e.g.* by the two conditions $\int |a(x)|(1+|x|)\,dx < \infty$ and $\operatorname{ess\,sup}_{x\in\mathbb{R}^d}(1+|x|)^{d+1}|a(x)| < \infty$. The smallness condition (12) could be be slightly relaxed, see [4].

2.2 The construction of the initial datum

This section devoted to a constructive proof of the following Lemma.

Lemma 1 *Let $d = 2,3$, and $\epsilon > 0$. Let also $N \in \mathbb{N}$ and $0 < t_1 < \cdots < t_N$ be a finite sequence. Then there exists a divergence-free vector field $a = (a_1, \ldots, a_d) \in \mathcal{S}(\mathbb{R}^d)$, such that*

$$\widetilde{a}(x) = a(\widetilde{x}), \qquad x \in \mathbb{R}^d \tag{13}$$

(see Section 1.3 for the notations) and such that the function $E(a)(t)\colon \mathbb{R}^+ \to \mathbb{R}$, defined by

$$E(a)(t) \equiv -\int_0^t \int e^{s\Delta}a_1(x)\, e^{s\Delta}a_2(x)\,dx\,ds, \tag{14}$$

changes sign inside $(t_i - \epsilon, t_i + \epsilon)$, for $i = 1, \ldots, N$.

Proof. It is convenient to separate the two and three-dimensional cases

The case $d = 2$. We start setting, for each $\alpha \in \mathbb{R}^2$,

$$\widehat{\psi}_\alpha(\xi) \equiv \widehat{\phi}(\xi - \alpha) + \widehat{\phi}(\xi + \alpha) - \widehat{\phi}(\xi - \widetilde{\alpha}) - \widehat{\phi}(\xi + \widetilde{\alpha}), \tag{15}$$

for some $\phi \in \mathcal{S}(\mathbb{R}^2)$ satisfying conditions (4). Next we introduce the divergence-free vector field $a_\alpha(x)$, through the relation

$$\widehat{a}_\alpha(\xi) = \begin{pmatrix} -i\xi_2\widehat{\psi}_\alpha(\xi) \\ i\xi_1\widehat{\psi}_\alpha(\xi) \end{pmatrix}. \tag{16}$$

Note that $a_\alpha \in \mathcal{S}(\mathbb{R}^2)$ is real-valued (because $\widehat{\psi}_\alpha$ is real-valued and such that $\widehat{\psi}_\alpha(\xi) = \widehat{\psi}_\alpha(-\xi)$) and satisfies the fundamental symmetry condition

$$\widetilde{a}_\alpha(x) = a_\alpha(\widetilde{x}). \tag{17}$$

Next we define $\widehat{\psi}^\delta$, a_α^δ as before, by simply replacing $\widehat{\phi}$ with $\widehat{\phi}^\delta$ in the corresponding definitions. The vector fields a_α^δ will be our "building blocks" of our initial datum.

Applying the Plancherel theorem to the right-hand side of (14) and using the symmetry relations $\widehat{\psi}_\alpha(\xi) = \widehat{\psi}_\alpha(-\xi)$, $|\widehat{\psi}_\alpha(\xi)| = |\widehat{\psi}_\alpha(\tilde{\xi})|$, we get

$$\begin{aligned} E(a_\alpha)(t) &= \int (1 - e^{-2t|\xi|^2}) \frac{\xi_1 \xi_2}{2|\xi|^2} |\widehat{\psi}_\alpha(\xi)|^2 \, d\xi \\ &= 2 \int_{\xi_1 \geq |\xi_2|} (1 - e^{-2t|\xi|^2}) \frac{\xi_1 \xi_2}{|\xi|^2} |\widehat{\psi}_\alpha(\xi)|^2 \, d\xi. \end{aligned} \tag{18}$$

From now on, the components of $\alpha \in \mathbb{R}^2$ will be assumed to satisfy the following conditions

$$\begin{cases} \alpha_1 > |\alpha_2| \\ \alpha_2 \neq 0. \end{cases} \tag{19}$$

This guarantees that for a sufficiently small $\delta > 0$ (*i.e.* when $\alpha_1 > |\alpha_2| + \delta\sqrt{2}$), we have

$$E(a_\alpha^\delta)(t) = 2 \int (1 - e^{-2t|\xi|^2}) \frac{\xi_1 \xi_2}{|\xi|^2} |\widehat{\phi}^\delta(\xi - \alpha)|^2 \, d\xi. \tag{20}$$

If we set

$$E_\alpha^{\text{app}}(t) \equiv \big(1 - e^{-2t|\alpha|^2}\big) \frac{\alpha_1 \alpha_2}{|\alpha|^2} \tag{21}$$

then we immediately obtain

$$E(a_\alpha^\delta)(t) \to E_\alpha^{\text{app}}(t), \qquad \text{as } \delta \to 0$$

uniformly with respect to $t \geq 0$.

We now associate to $(t_1, \ldots, t_N)$ two more sequences (to be chosen later) $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}_+^{N+1}$ and $(\alpha_1, \ldots, \alpha_{N+1}) \in \mathbb{R}^{2(N+1)}$, where $\alpha_j = (\alpha_{j,1}, \alpha_{j,2})$. First we require that the components $\alpha_{j,1}$ and $\alpha_{j,2}$ satisfy condition (19) for all $j = 1, \ldots, N+1$ and that $\alpha_j \neq \alpha_{j'}$, for $j \neq j'$ and $j, j' = 1, \ldots, N+1$. This second requirement ensures that the supports of $\widehat{a}_{\alpha_j}^\delta$ and $\widehat{a}_{\alpha_{j'}}^\delta$ are disjoint when δ is sufficiently small.

We now consider the initial data of the form

$$a^\delta(x) \equiv \sum_{j=1}^{N+1} \lambda_j a_{\alpha_j}^\delta(x). \tag{22}$$

Owing to the condition on the supports of $\widehat{a}_{\alpha_j}^\delta$, we see that for $\delta > 0$ small enough,

$$E(a^\delta)(t) = \sum_{j=1}^{N+1} \lambda_j^2 E(a_{\alpha_j}^\delta)(t). \tag{23}$$

Let

$$\mu_j = \frac{\lambda_j^2 \, \alpha_{j,1} \, \alpha_{j,2}}{|\alpha_j|^2} \qquad \text{and} \qquad A_j = e^{-2|\alpha_j|^2}. \tag{24}$$

Thus, as $\delta \to 0$, we get $E(a^\delta)(t) \to E^{\text{app}}(t)$, uniformly in $[0, \infty)$, where

$$E^{\text{app}}(t) = \sum_{j=1}^{N+1} \mu_j \big(1 - A_j^t\big). \tag{25}$$

Let us observe that

$$\frac{\mathrm{d}E^{\mathrm{app}}}{\mathrm{d}t}(t) = -\sum_{j=1}^{N+1} \mu_j \, \log(A_j) A_j^t. \tag{26}$$

We want to determine $(\lambda_1, \dots, \lambda_{N+1})$ and $(\alpha_1, \dots, \alpha_{N+1})$ in a such way that $E^{\mathrm{app}}(t)$ vanishes at $t_1, \dots, t_N$, and changing sign at those points. This leads us to study the system of N equalities and N 'non-equalities',

$$\begin{cases} E^{\mathrm{app}}(t_i) = 0 \\ \dfrac{\mathrm{d}E^{\mathrm{app}}}{\mathrm{d}t}(t_i) \neq 0. \end{cases} \qquad i = 1, \dots, N. \tag{27}$$

Let us choose $|\alpha_j|^2 = \gamma j$, for an arbitrary $\gamma > 0$ (this choice is not essential, but will greatly simplify the calculations) and set $T_i = e^{-2\gamma t_i}$. Recalling (24), we get $A_j^{t_i} = T_i^j$ and $\log(A_j) = -2\gamma j$. In order to study the system (27), we introduce the $(N+1)^2$-matrix

$$M := \begin{pmatrix} 1 - T_1 & 1 - T_1^2 & \cdots & 1 - T_1^{N+1} \\ \vdots & \vdots & & \vdots \\ 1 - T_N & 1 - T_N^2 & \cdots & 1 - T_N^{N+1} \\ T_1 & 2T_1^2 & \cdots & (N+1)T_1^{N+1} \end{pmatrix} \tag{28}$$

We claim that $\det M \neq 0$. Indeed, by an explicit computation,

$$\det(M) = -T_1 \, (1 - T_1) \prod_{i=1}^{N} (1 - T_i) \prod_{i=2}^{N} (T_1 - T_i) \prod_{1 \le i < i' \le N} (T_{i'} - T_i).$$

Recalling that $T_i = e^{-2\gamma t_i} \in (0, 1)$ and that $t_i \neq t_{i'}$ proves our claim. The above formula can be checked by induction. Otherwise, one can reduce M after elementary factorizations to a Vandermonde-type matrix (see [10] for explicit formulae on determinants).

Then, for any $c \neq 0$, the linear system with unknown $\boldsymbol{\mu} = (\mu_1, \dots, \mu_{N+1})$,

$$M\boldsymbol{\mu} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c \end{pmatrix} \tag{29}$$

has a unique solution $\boldsymbol{\mu}^* \in \mathbb{R}^{N+1}$, $\boldsymbol{\mu}^* \neq 0$. By our construction, the function $E^{\mathrm{app}}(t)$ obtained taking $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ satisfies the N-equations and the first 'non-equality' of the system (27). More precisely, we get $(\mathrm{d}/\mathrm{d}t)E^{\mathrm{app}}(t_1) = c/(2\gamma) \neq 0$. The other $N-1$ 'non-equalities' of the system (27) are then automatically fulfilled. Indeed if, otherwise, we had $(\mathrm{d}/\mathrm{d}t)E^{\mathrm{app}}(t_i) = 0$, for some $i = 2, \dots, N$, then the matrix obtained replacing in (28) the last line with

$$T_i \quad 2T_i^2 \quad \cdots \quad (N+1)T_i^{N+1}$$

would have been of determinant zero, thus contredicting our preceding formula for $\det(M)$.

By conditions (24), for all $j = 1, \dots, N+1$, the real number μ_j^* defines (in a non-unique way) a real λ_j^* and a vector $(\alpha_{j,1}^*, \alpha_{j,2}^*)$ with components satisfying (19), such that $(\lambda_1, \dots, \lambda_{N+1}) \neq (0, \dots, 0)$.

We now consider the initial data a_*^δ obtained from formula (22), choosing $\lambda_j = \lambda_j^*$ and $\alpha_j = \alpha_j^*$ for $j = 1, \dots, N+1$. With this choice, the corresponding function $E^{\rm app}(t)$ changes sign in each neighborhood of t_i.

By the uniform convergence of $E(a_*^\delta)(t)$ to $E^{\rm app}(t)$ as $\delta \to 0$, we see that if $\delta > 0$ is small enough then $E(a_*^\delta)$ changes sign in the interval $(t_i - \epsilon, t_i + \epsilon)$, for $i = 1, \dots, N$.

The conclusion of Lemma 1 in the two dimensional case now follows.

The case $d = 3$. We only indicate the modifications that have to be done to the above arguments. Let us set, for $\alpha \in \mathbb{R}^3$,

$$\widehat{a}_\alpha(\xi) = \begin{pmatrix} (i\xi_2 - i\xi_3)\widehat{\psi}_\alpha(\xi) \\ (i\xi_3 - i\xi_1)\widehat{\psi}_\alpha(\xi) \\ (i\xi_1 - i\xi_2)\widehat{\psi}_\alpha(\xi) \end{pmatrix} \tag{30}$$

where ψ is defined by

$$\begin{aligned}\widehat{\psi}_\alpha(\xi) =& \widehat{\phi}(\xi - \alpha) + \widehat{\phi}(\xi - \widetilde{\alpha}) + \widehat{\phi}(\xi - \widetilde{\widetilde{\alpha}}) \\ &+ \widehat{\phi}(\xi + \alpha) + \widehat{\phi}(\xi + \widetilde{\alpha}) + \widehat{\phi}(\xi + \widetilde{\widetilde{\alpha}}).\end{aligned}$$

In this way, $a \in \mathcal{S}(\mathbb{R}^3)$ is a real valued divergence-free vector field with the rotational symmetry (13). As before, we can define also the rescaled vector field a^δ, for any $\delta > 0$.

The conditions to be imposed on the components of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are now

$$\min(\alpha_2, \alpha_3) > \alpha_1^+, \qquad \alpha_2 \neq \alpha_3, \tag{31}$$

where $\alpha_1^+ = \max(\alpha_1, 0)$. Geometrically, the inequality in (31) corresponds to cutting $\mathbb{R}^3$ into six congruent regions, that can be obtained from each other through the orthogonal transforms $\alpha \mapsto \widetilde{\alpha}$ and $\alpha \mapsto -\alpha$, and then selecting one of these regions. If $\delta > 0$ is small enough, in a such way that $\alpha + B(0, \delta)$ is contained in the region $\Gamma \subset \mathbb{R}^3$ defined by (31), then we get, recalling the definition (14) of $E(a_\alpha)$, the three-dimensional counterpart of (20):

$$E(a_\alpha)(t) = 3 \int (1 - e^{-2t|\xi|^2}) \frac{(\xi_1 - \xi_3)(\xi_2 - \xi_3)}{|\xi|^2} |\widehat{\phi}^\delta(\xi - \alpha)|^2 \, d\xi.$$

(Here one first applies Plancherel Theorem, then the integral over $\xi \in \mathbb{R}^3$ is rewritten, because of the symmetries of $\widehat{a}_\alpha$, as $6 \int_\Gamma \dots \, d\xi$). We have, for all $t \geq 0$ and as $\delta \to 0$,

$$E(a_\alpha^\delta)(t) \to E_\alpha^{\rm app}(t) \equiv (1 - e^{-2t|\alpha|^2}) \frac{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}{|\alpha|^2}.$$

For $\alpha_j = (\alpha_{j,1}, \alpha_{j,2}, \alpha_{j,3}) \in \mathbb{R}^3$, $j = 1, \dots, N+1$, with components satisfying (31), we now set

$$\mu_j = \frac{\lambda_j^2 \, (\alpha_{j,1} - \alpha_{j,3})(\alpha_{j,2} - \alpha_{j,3})}{|\alpha_j|^2}. \tag{32}$$

As in the two-dimensional case, it is possible to choose the phases α_j in a such way that $|\alpha_j|^2 = \gamma j$, where $\gamma > 0$ is arbitrary. This leaves unchanged the definitions of A_j and T_i.

Now solving, as before, the linear system (29), shows that it is possible to construct an initial datum of the form

$$a^\delta = \sum_{j=1}^{N+1} \lambda_j a_{\alpha_j}^\delta,$$

such that $E(a)(t)$ has a non-constant sign inside the intervals $(t_i-\epsilon,t_i+\epsilon)$, for $i=1,\ldots,N$.

Lemma 1 is now established.

$\square$

2.3 End of the proof of Theorem 1

We are now in position to deduce from Lemma 1 and the facts recalled in Section 2.1 the conclusion of Theorem 1.

Step 1. *Constructing a solution such that* $t\mapsto \int_0^t\!\!\int u_1u_2(x,s)\,dx\,ds$ *changes sign near* t_1, $t_2,\ldots,t_N$.

Let us consider the initial datum a constructed in Lemma 1. If necessary, we modify a by multipliying it by a small constant $\eta_0>0$ in order to ensure that the corresponding solution u of the Navier–Stokes system is defined globally in time. Without loss of generality, we can and do assume $\eta_0=1$.

With the same notation of Section 2.1, let $K(a)\colon \mathbb{R}^+\to\mathbb{R}^{d\times d}$,

$$K(a)(t)\equiv\int_0^t\!\!\int\big(\Phi(a)\otimes\Phi(a)\big)(x,s)\,dx\,ds. \tag{33}$$

Let $\mathcal{F}$ be the space defined in Section 2.1, and normed by (11). For $\eta>0$ small enough we can write, by expansion (10),

$$K(\eta\,a)(t)=\sum_{k=2}^{\infty}\eta^k\,S_k(e^{t\Delta}a)(t) \tag{34}$$

where $S_k\colon\mathcal{F}\to\mathbb{R}^{d\times d}$ is the restriction to the diagonal of $\mathcal{F}^k=\mathcal{F}\times\cdots\times\mathcal{F}$ of a k-multilinear operator defined from $\mathcal{F}^k$ to $\mathbb{R}^{d\times d}$.

The embedding of $\mathcal{F}\subset\mathcal{C}([0,\infty),L^2(\mathbb{R}^d))$ implies that, for all $t>0$, the series in (34) is absolutely convergent. But,

$$S_2(e^{t\Delta}a)(t)=\int_0^t\!\!\int\big(e^{s\Delta}a\otimes e^{s\Delta}a\big)(x,s)\,dx\,ds,$$

so that, according to the notations of Lemma 1,

$$\big(S_2(a)\big)_{1,2}(t)=-E(a)(t).$$

For any time t such that $E(a)(t)>0$ (respectively, $E(a)(t)<0$), we can find a small $\eta_t>0$ such that the solution $\Phi(\eta\,a)$ of the Navier–Stokes system starting from $\eta\,a$ satisfies, for all $0<\eta<\eta_t$,

$$K(\eta\,a)_{1,2}(t)<0,\qquad \text{(respectively, } K(\eta\,a)_{1,2}(t)>0).$$

By Lemma 1, this observation can be applied for t belonging to two suitable finite sequences of times (that can be taken arbitrarily close to $(t_1,\ldots,t_N)$ in the $\mathbb{R}^N$-norm). We conclude that if $\eta>0$ is small enough then $K(\eta\,a)_{1,2}(t)$ changes sign in $(t_i-\epsilon,t_i+\epsilon)$, for $i=1,\ldots,N$. In particular, because of the continuity of the map $t\mapsto K(\eta\,a)(t)$, there is a point t_i^* inside each one of these intervals where $K(\eta\,a)_{1,2}(t_i^*)=0$.

Step 2. *Analysis of the orthogonality relations* $\int_0^t\!\!\int(u_ju_k)(x,s)\,dx\,ds=c(t)\delta_{j,k}$.

Let $u=\Phi(\eta\, a)$ be the solution constructed in the first step. Notice that $\widetilde{u}(x,t)=u(\widetilde{x},t)$ for all $t\ge 0$, *i.e.*, the condition $\widetilde{a}(x)=a(\widetilde{x})$ propagates during the evolution. Indeed, this is simple consequence of the invariance of the Navier–Stokes equations under the tranformations of the orthogonal group $O(d)$ and the uniqueness of strong solutions. Thus, $\int u_1^2\,dx = \int u_2^2\,dx$. When $d=3$, such integrals equal, of course, $\int u_3^2\,dx$, and we have also $\int u_1u_2\,dx = \int u_2u_3\,dx = \int u_3u_1\,dx$. We deduce that, for all $t\ge 0$ (we denote here by I the $d\times d$ identity matrix):

$$\exists\, c(t):\ K(\eta\, a)(t)=c(t)I \qquad \text{if and only if} \qquad K(\eta\, a)_{1,2}(t)=0. \tag{35}$$

(This equivalence is no longer valid for $d\ge 4$. For a proof of the theorem in the higher dimensional case one should consider flows invariant under larger discrete subgroups of $O(d)$).

Step 3. *The far-field asymptotics of the velocity field.*

By the result of [4] (see Theorem 1.2 or Theorem 1.7, applied in the particular case of the datum $\eta\, a\in\mathcal{S}(\mathbb{R}^d)$) we know that, for all $t>0$,

$$u(x,t)=\nabla_x\Pi(x,t)+\mathcal{O}_t(|x|^{-d-2}), \qquad \text{as } |x|\to\infty, \tag{36}$$

where,

$$\Pi(x,t)=\gamma_d\sum_{h,k}\left(\frac{\delta_{h,k}}{d|x|^d}-\frac{x_hx_k}{|x|^{d+2}}\right)\cdot K(\eta\, a)_{h,k}(t) \tag{37}$$

and $\gamma_d\neq 0$ is a constant.

Concerning the first term $\nabla_x\Pi$ on the right-hand side of (36), for each fixed $t>0$ two situations can occur. Either the function $x\mapsto\nabla_x\Pi(x,t)$ is identically zero, or this function is homogeneous of degree exactly $-d-1$. But, for all fixed $t>0$ (see [4], Proposition 1.6),

$$\nabla_x\Pi(\cdot,t)\equiv 0 \qquad \text{if and only if} \qquad \exists\, c(t)\in\mathbb{R}:\ K(\eta\, a)_{h,k}(t)=c(t)\delta_{h,k}. \tag{38}$$

Combining conditions (35)-(38) with the asymptotic profile (36), we deduce from the analysis we made in Step 1, the upper bound

$$|u(x,t_i^*)|\le C|x|^{-d-2}, \qquad i=1,\ldots,N$$

and the lower bound,

$$|u_j(x,t_i')|\ge c_\omega|x|^{-d-1}, \qquad i=1,\ldots,N, \quad j=1,\ldots,d,$$

for all $|x|$ large enough and some points t_i' distant less than ϵ from t_i. Here, $C>0$ is independent on x and c_ω is independent on $|x|$, but will depend on the projection $\omega=\frac{x}{|x|}$ of x on the sphere $\mathbb{S}^{d-1}$. In fact, we can take $c_\omega>0$, unless $\partial_{x_j}\Pi(\omega,t_i')$ has a zero at the point $\omega\in\mathbb{S}^{d-1}$. But one deduces from (37) that the zeros of $\partial_{x_j}\Pi(\omega,t_i')$ are exactly the zeros on the unit sphere of a homogeneous polynomial of degree three. Therefore, $c_\omega>0$ for almost every $\omega\in\mathbb{S}^{d-1}$.

Theorem 1 is now established.

□

2.4 Explicit examples

Let us exhibit an explicit example of our construction, in the simplest case $N = 1$. We set, for $d = 2$, and a function ϕ satisfying conditions (4),

$$a(x) = \eta \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Big[\phi^\delta(x)\Big(\sqrt{3}\cos(\sqrt{3}x_1 + x_2) - \sqrt{3}\cos(x_1 + \sqrt{3}x_2) \\ + \sqrt{2}\cos(\sqrt{6}x_1 - \sqrt{2}x_2) - \sqrt{2}\cos(-\sqrt{2}x_1 + \sqrt{6}x_2)\Big)\Big].$$

This expression is obtained taking, in Lemma 1, $T_1 = \frac{1}{2}$, $\alpha_1 = (\sqrt{3}, 1)$, $\alpha_2 = (\sqrt{6}, -\sqrt{2})$ and $\gamma = 4$, and observing that $\lambda_1^2 = \frac{3}{2}\lambda_2^2$. If $|\eta|$ and δ are positive and small enough, then the solution $u(x,t)$ of Navier–Stokes starting from a concentrates/diffuses when $t \simeq \frac{1}{8}\log(2)$.

Three-dimensional examples are obtained in a very similar way: choose again $T_1 = \frac{1}{2}$, next $\alpha_1 = (0, 1, \sqrt{3})$, $\alpha_2 = (0, \sqrt{6}, \sqrt{2})$. The relation between the coefficients λ_1 and λ_2 is now $\lambda_1^2 = \frac{\sqrt{3}}{2}\lambda_2^2$. This leads us to introduce the function

$$f^\delta(x) \equiv \phi^\delta(x)\Big[3^{1/4}\Big(\cos(x_2 + \sqrt{3}x_3) + \cos(x_1 + \sqrt{3}x_2) + \cos(\sqrt{3}x_1 + x_3)\Big) \\ + \sqrt{2}\Big(\cos(\sqrt{6}x_2 + \sqrt{2}x_3) + \cos(\sqrt{6}x_1 + \sqrt{2}x_2) + \cos(\sqrt{2}x_1 + \sqrt{6}x_3)\Big)\Big], \tag{39a}$$

where $\delta > 0$ and ϕ satisfies conditions (4). If a is the vector field

$$a = \eta\,\mathrm{curl}(f^\delta, f^\delta, f^\delta), \tag{39b}$$

with $|\eta|, \delta > 0$ small enough, then the solution arising from a concentrates/diffuses, as before, around $t \simeq \frac{1}{8}\log(2)$.

Remark 2 The smallness of η was important for the applicability of our analyticity argument. However, it would be interesting to know whether three-dimensional data as those constructed in (39a)-(39b), with *large* coefficients η, still feature some kind of concentration effects in finite time.

3 Linear concentration-diffusion effects

In this section we would like to give an elementary example of a Navier–Stokes flow featuring a concentration effect of a quite different nature. A similar effect can be observed if we replace the Navier–Stokes nonlinearity by a more general term (not necessarily quadratic). Let us focus, for example, on the three-dimensional case.

Consider the two initial data in $\mathbb{R}^3$ (here $\eta \neq 0$ is a constant)

$$\dot{a}(x) = \eta\Big(-\partial_2\big[\log(e + |x|^2)^{-1}\big],\, \partial_1\big[\log(e + |x|^2)^{-1}\big],\, 0\Big) \tag{40}$$

and

$$\overline{a}(x) = \dot{a}(x)\sin(|x|^2). \tag{41}$$

In both cases, a direct computation shows that such data are divergence-free and belong to $L^3(\mathbb{R}^3)$, but not to $L^p(\mathbb{R}^3)$ for $1 \le p < 3$.

In his well-known paper [9], Kato could obtain, for a solution u of the Navier–Stokes system emanating from a, local and global results implying $u(t) \in L^q(\mathbb{R}^3)$ for $t > 0$, with $q \ge 3$, under the assumption $a \in L^3(\mathbb{R}^3)$. He also obtained results implying $u(t) \in L^p \cap L^q(\mathbb{R}^3)$, always for $q \ge 3$, under the stronger assumption $a \in L^p \cap L^3(\mathbb{R}^3)$, with $1 \le p < 3$. Immediately after stating his theorems, Kato observed the following

Remark 3 (see [9]) "The spatial decay expressed by the property $u(t) \in L^q(\mathbb{R}^3)$ is of interest. Note that q is restricted to $q \ge 3$. *We were able to give no results for* $q < 3$ *under the* [only] *assumption* $a \in L^3(\mathbb{R}^3)$".

The solutions arising from the data $\dot{a}$ and $\overline{a}$ (both uniquely defined in $\mathcal{C}([0,T), L^3(\mathbb{R}^3))$, for some $0 < T \le \infty$ *a priori* depending on η) show that when one assumes only $a \in L^3(\mathbb{R}^3)$ then "everything can happen". Indeed, it is not difficult to prove the following claims:

1. The solution $u(t)$ arising from the datum $\dot{a}$ *does not belong to* $L^q(\mathbb{R}^3)$, for any $t \in (0,T)$ and any $q \in [1,3)$.
2. The solution $u(t)$ arising from the datum $\overline{a}$ *does belong to* $L^q(\mathbb{R}^3)$ for all $t \in (0,T)$ and all $q \in [1,3)$.

Therefore, in the latter case, the solution enjoys some kind of spatial concentration effect. However this effect does not rely on special geometric features of the flow, but only on the oscillatory character of $\overline{a}$.

The first claim is immediate, because whenever $|a(x)| \le C(1+|x|)^{-1}$ and $|\nabla a(x)| \le C(1+|x|)^{-2}$, then arguing as in [5, Sec. 4] one obtains for $t > 0$ an estimate of the form $|e^{t\Delta}a(x) - a(x)| \le C(t)(1+|x|)^{-2}$. Then, from the equation $u = e^{t\Delta}a + B(u,u)$ (the notation is as in Section 2.1), the estimate $|u(x,t) - a(x)| \le \bar{C}(t)(1+|x|)^{-2}$. Applying this observation to $\dot{a}$ yields the conclusion.

Let us sketch the proof of our second claim. The faster and faster oscillations of $\overline{a}$ imply that, for $t > 0$ and $|x|$ large enough,

$$|e^{t\Delta}\overline{a}(x)| \le C_m(t)|x|^{-m-1}\log^{-2}|x|, \qquad m = 0,1,2\dots$$

where the functions $C_m(t)$ are locally bounded in $(0,\infty)$ and satisfy $C_m(t) \sim t^{-m/2}$ as $t \to 0$. In particular, $\|e^{t\Delta}\overline{a}\|_{3/2} \le C_1(t)$. Let us iterate the integral Navier–Stokes equation:

$$u(t) = e^{t\Delta}\overline{a} + B(e^{t\Delta}\overline{a}, e^{t\Delta}\overline{a}) + 2B(e^{t\Delta}\overline{a}, B(u,u)) + B(B(u,u), B(u,u)). \tag{42}$$

Now applying elementary Hölder and Young inequalities in the right-hand side of (42) (we can freely use here that $\|u(t)\|_3$ is bounded in $(0,T)$), we get, for all $t \in (0,T)$, $\|u(t)\|_1 \le C(t)$ with $C(t) \sim t^{-1}$ as $t \to 0$. By interpolation one obtains that $u(t) \in L^q(\mathbb{R}^3)$ in $(0,T)$ for $1 \le p < 3$, with an estimate on the blow-up of the L^q-norm as $t \to 0$.

Let us observe that for $0 < t < T$, even though $u(t)$ decays pointwise much faster than $\overline{a}$, as $|x| \to \infty$, the decay of $u(t)$ does not exceed that of $|x|^{-4}$, accordingly with the limitations on the spatial localization described in the introduction.

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